CHAPTER 6

STRUTS

6.0 Introduction

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. **The buckling occurs owing to one the following reasons.**

(a) the strut may not be perfectly straight initially.

(b) the load may not be applied exactly along the axis of the Strut.

(c) one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

Failure of a column

The failure of a column takes place due to one of the following stresses set up in the columns
(i) Direct compressive stresses,
(ii) Buckling stresses, and
(iii) Combined of direct compressive and buckling stresses.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should then be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is the quantity I may be written

as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

is called the **slenderness ratio**. It's numerical value indicates whether the member falls into the class of columns or struts.

6.1. Euler's Theory

Assumptions made in the Euler's column theory

The following assumptions are made in the Euler's column theory:

1. the column is initially perfectly straight and the load is applied axially.
2. the cross-section of the column is uniform throughout its length.
3. the column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

Sign Conventions

The following sign conventions for the bending of the columns will be used

1. A moment which will bend the column with its convexity towards its initial central line as shown in Fig. 1(a) is taken as positive. In Fig. 1(a), AB represents the initial centre line of a column. Whether the column bends taking the shape AB' or AB", the moment producing this type of curvature is positive.

2. A moment which will tend to bend the column with its concavity towards its initial centre line as shown in Fig. 1(b) is taken as negative.

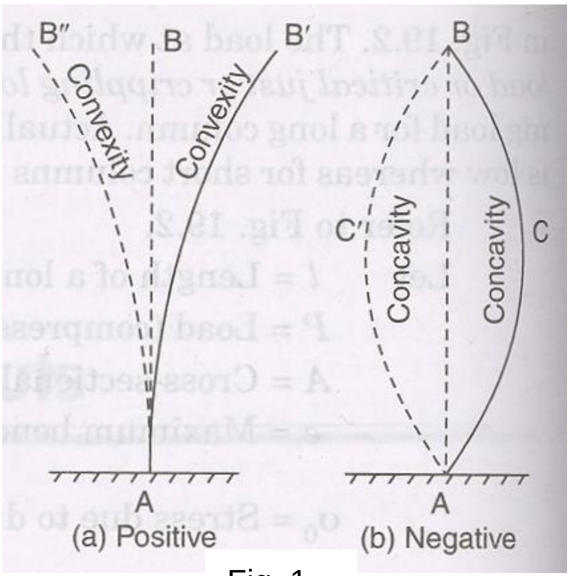


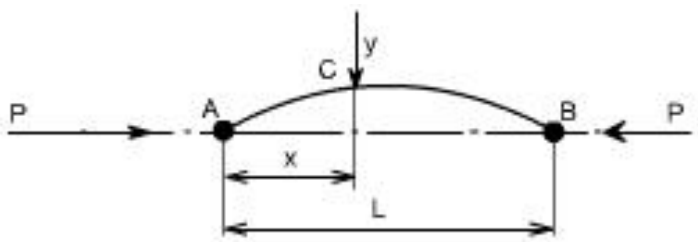
Fig. 1

The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

Consider an axially loaded strut, shown below, and is subjected to an axial load P; This load P produces a deflection y at a distance x from one end.

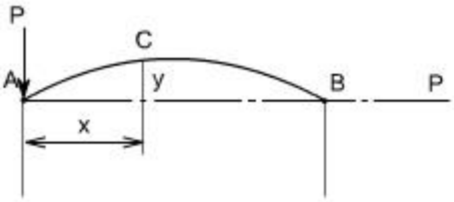
Assume that the ends are either pin jointed or rounded so that there is no moment at either

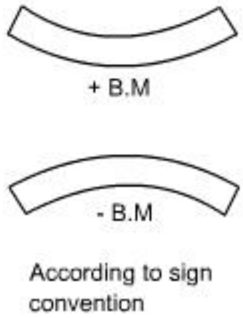


end.

Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.





$$B.M|_C = -Py$$

Futher,we know that

$$EI\,\frac{d^2y}{dx^2} = M$$

$$EI\,\frac{d^2y}{dx^2} = -P.y = M$$

This equation of M is not a function x. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\,\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in y but we can't say at this stage where the deflection would be maximum or minimum. So the above differential equation can be arranged in the following form

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

Let us define an operator

$$D = d/dx$$

$$(D^2 + n^2)\,y = 0\ ,\ \text{where } n^2 = P/EI$$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0].

Thus y = A cos (nx) + B sin (nx)

Where A and B are some constants.

Therefore $$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0.

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either B = 0, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0, that y0 for all values of x hence the strut has not buckled yet. Therefore, the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \text{ or } \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

Applying the second boundary condition gives

From the above relationship the least value of P which will cause the strut to buckle called

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin\ nL = 0$$

the **Euler Crippling Load** P_e from which we obtain:

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL =0; the strut will remain perfectly straight since

y = B sin nL = 0

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

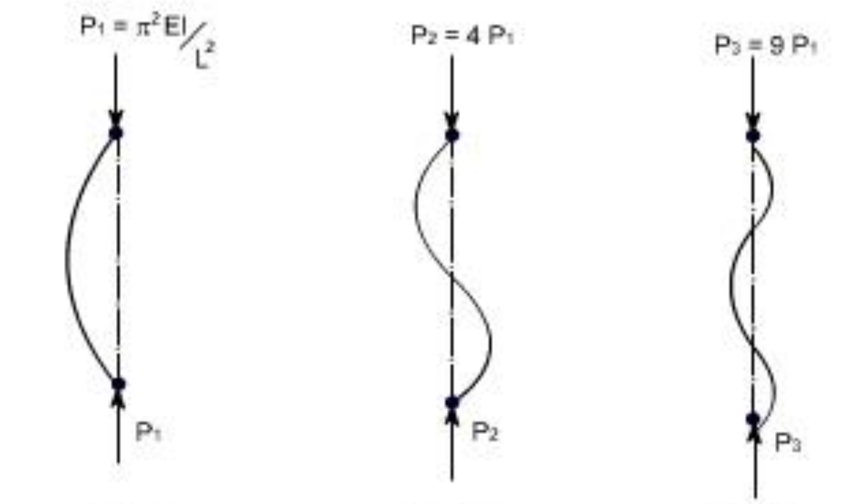
$$\text{Therefore } \; n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \; \sin \frac{\pi x}{L}$$

Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that L remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; likewise it will be found that the maximum stress is not proportional to load.

The solution chosen of nL = □ is just one particular solution; the solutions nL= 2□, 3□, 5□ etc are equally valid mathematically and they do, infact, produce values of $P_e$which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.



nL = π Fundamental Mode (First harmonic)

nL = 2π Second harmonic (mid point bracing)

nL = 3π Third harmonic (Third point bracing)

The value selected above is so

called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution $nL = 2\pi$ produces buckling in two half waves, 3π in three half-waves etc.

$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

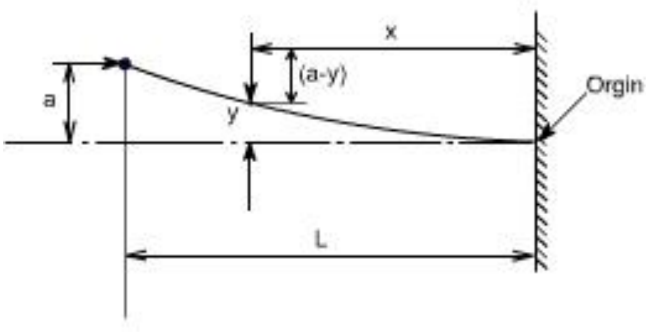
$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \quad \text{or} \quad P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

6.2 <u>Struts and columns with other end conditions:</u> Let us consider the struts and columns having different end conditions

Case B: One end fixed and the other free:

Writing down the value of bending moment at the point C



$$B.M|_c = P(a - y)$$

Hence, the differential equation becomes,

$$EI\frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

$$\text{Let } \frac{P}{EI} = n^2$$

Hence in operator form, the differential equation reduces to $(D^2 + n^2)\, y = n^2 a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

$$y_{gen} = A\cos(nx) + \sin(nx) + P.I$$

where P.I = a particular value of y which satisfies the differential equation

Hence $y_{P.I} = a$

Therefore the complete solution becomes

$$y = A\cos(nx) + B\sin(nx) + a$$

Now imposing the boundary conditions to evaluate the constants A and B

(i) at $x = 0$; $y = 0$; This yields $A = -a$

(ii) at $x = 0$; $dy/dx = 0$; This yields $B = 0$

Hence $y =$ □$a\cos(nx) + a$

Futher, at $x = L$; $y = a$

Therefore $a = -a\cos(nx) + a$ or $0 = \cos(nL)$

Now the fundamental mode of buckling in this case would be

$$nL = \frac{\pi}{2}$$

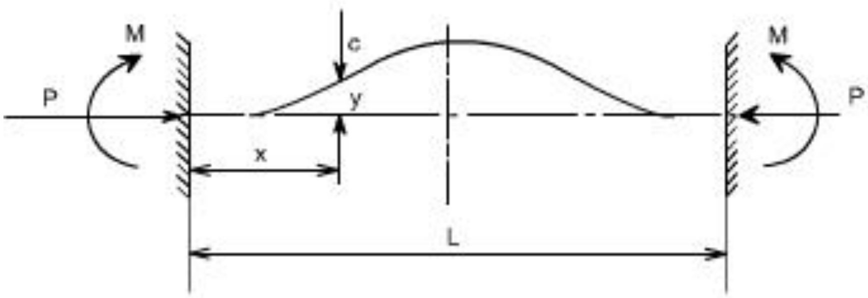
$$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}, \text{Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case C: Strut with fixed ends:

Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

Bending Moment at point C = P.y



Thus,

$$EI\frac{d^2y}{dx^2} = M - Py$$

or $\frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\,integral}$$

$$y|_{P.I} = \frac{M}{n^2EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\,Cosnx + A\,Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0;y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\,Cos\,nx + \frac{M}{P}$$

$$y = \frac{M}{P}\,(1 - Cos\,nx)$$

Futher, it may be noted that at $x = L; y = 0$

Then $0 = \frac{M}{P}\,(1 - Cos\,nL)$

Thus, either $\frac{M}{P} = 0$ or $(1 - Cos\,nL) = 0$

obviously, $(1 - Cos\,nL) = 0$

$\cos nL = 1$

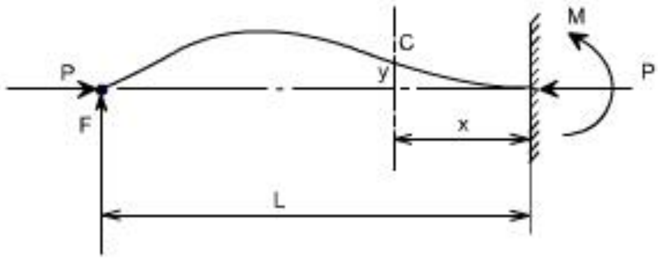
Hence the least solution would be

$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\,L = 2\pi$, Thus, the buckling load or crippling load is

$$P_e = \frac{4\pi^2.EI}{L^2}$$

Case d: **One end fixed, the other pinned**



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B.M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L-x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L-x)$$

$$y_{particular} = \frac{F}{n^2EI}(L-x) \text{ or } y = \frac{F}{P}(L-x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L-x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L-x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L-x)\right]$$

Also when x = L ; y = 0

Therefore nL Cos nL = Sin nL   or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\, L = 4.49$$

$$\frac{P_e}{EI} L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\, EI}{L^2}$$

6.3. <u>Equivalent Strut Length</u>

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

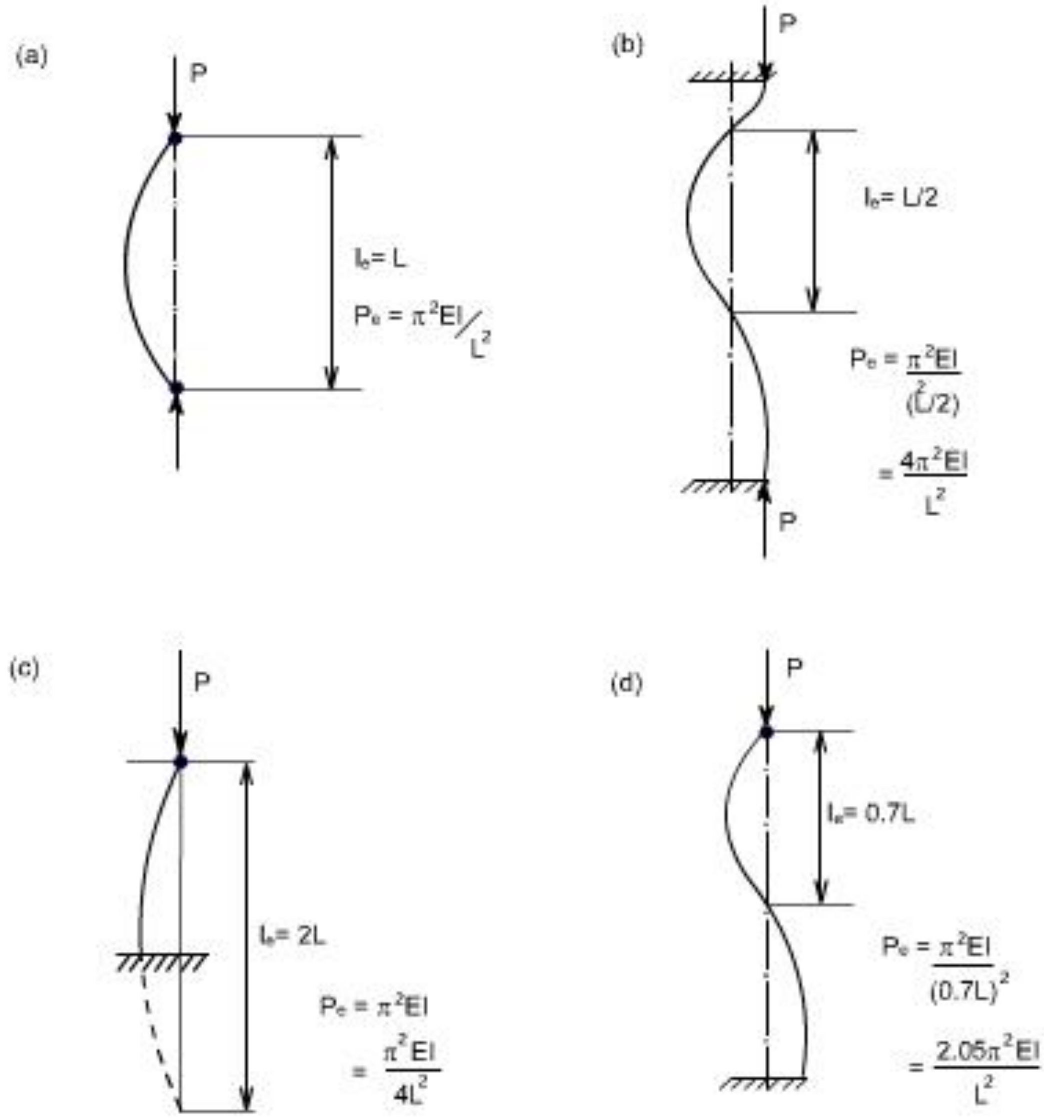
The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the free body diagram would indicate that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

(a) Both ends pinned (c) One end fixed, other free

(b) Both ends fixed (d) One end fixed and other pinned



S.No.	End conditions of column	Crippling load in terms of		Relation between effective length and actual length
		Actual length	Effective length	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

Table

1: Relation between effective and actual length

There are two values of moment of inertia i.e., I_{xx} and I_{yy}. The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

EXAMPLES

1. A solid round bar 3 m long and 5 cm in diameter is used as a strut with both ends hinged. Determine the crippling (or collapsing) load. Take E = 2.0 x 10^5 N/mm^2.

Sol. Given:

Length of bar, l = 3 m = 3000 mm

Diameter of bar, d = 5 cm = 50 mm

Young's modulus, E = 2.0 x 10^5 N/mm^2

Moment of inertia, $I = \frac{\pi}{64} \times 5^4 = 30.68 cm^4 = 30.68 \times 10^4 mm^4$

Let P = Crippling load.

As both the ends of the bar are hinged, hence the crippling load is given by equation

$$\frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2\times 10^5 \times 30.68\times 10^4}{3000^2} = 67288 \text{ N} = 67.288 \text{ kN. Ans.}$$

2. For the problem 1 determine the crippling load, when the given strut is used with the following conditions

(i) One end of the strut is fixed and the other end is free

(ii) Both the ends of strut are fixed

(iii) One end is fixed and other is hinged.

Sol. Given:

The data from Problem 1, is l = 3000 mm, diameter = 50 mm, E = 2.0 x 10^5 N/mm^2 and I = 30.68 x 10^4 mm^4.

Let P = Crippling load.

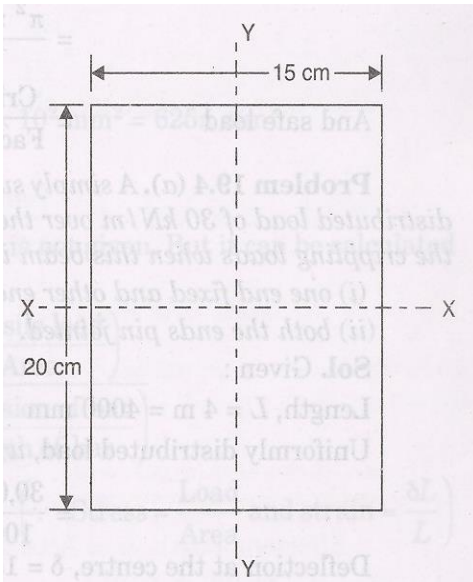
(i) Crippling load when one end is fixed and other is free

We know that (w.k.t.) $P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 \times 2\times 10^5 \times 30.68\times 10^4}{4\times 3000^2} = 16822\text{N}$ Ans.

3. A column of timber section 15cm x 20 cm is 6 metre long both ends beingfixed. If the Young's modulus for timber = 17.5 kN/mm^2, determine:

(i) Crippling load and

(ii) Safe load for the column if factor of safety = 3.



Sol. Given:

Dimension of section = 15 cm x 20 cm

Actual length, l = 6 m = 6000 mm

Young's modulus, E = 17.5 kN/mm^2

(i) Let P = Crippling load

Using equation below, we get

w.k.t. $P = \frac{\pi^2 EI}{L_e^2}$ …(i) where L_e = effective length = $\frac{l}{2}$ (when both the ends fixed)

$$= \frac{6000}{2} = 3000mm \qquad (\because l = 6000mm)$$

l = Least value of moment of inertia

Moment of inertia of the section about X-X axis,

$$I_{xx} = \frac{15 \times 20^3}{12} 10000cm^4 = 10000 \text{ x} 10^4 \text{ mm}^4$$

And moment of inertia of the section about Y-Yaxis,

$$I_{yy} = \frac{20 \times 15^3}{12} = 5625cm^4 = 5625 \times 10^4 mm^4.$$

Since I_{yy} is less than I_{xx}, therefore the column

will tend to buckle in Y-Y direction.

And the value of I will be the least value of the two moment of inertia.

I=5625cm^4= 5625 x 10^4 mm^4

Substituting the values of I = 5625 x 10 mm^4 and L = 3000 mm in equation (i), we get

`

$$P = \frac{\pi^2 \times 17.5 \times 5625 \times 10^4}{3000} = 1079.48kN \qquad \text{Ans}$$

(ii) Safe load for the column

Factor of safety = 3.0 (given)

$$\text{Safe load} = \frac{Crippling\ load}{Factor\ of\ safety} = \frac{1079.48}{3} = 359.8\ say 360\ kN \qquad \text{Ans.}$$

4. A simply supported beam of length 4 metre is subjected to a uniformly distributed load of 30 kN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions:

(i) one end fixed and other end hinged

(ii) both the ends pin jointed.

Sol. Given:

Length, L =4 m = 4000 mm

Uniformly distributed load, w = 30 kN/m = 30,000 N/m

$$= \frac{30{,}000}{1000} \text{ N/mm} = 30 \text{ N/mm}$$

Deflection at the centre, $\delta = 15$ mm.

For a simply supported beam, carrying U.D.L. over the whole span, the deflection at the centre is given by,

$$\delta = \frac{5}{384} \times \frac{w \times L^4}{EI}$$

or

$$15 = \frac{5}{384} \times \frac{30 \times 4000^4}{EI}$$

$$\therefore \quad EI = \frac{5}{384} \times \frac{30 \times 4000^4}{15}$$

$$= \frac{5}{384} \times \frac{3 \times 256}{15} \times 10^{13} = \frac{2}{3} \times 10^{13} \text{ N mm}^2.$$

(i) Crippling load when the beam is used as a column with one end fixed and other end hinged.

The crippling load P for this case in terms of actual length is given by equation (19.4) as

$$P = \frac{2\pi^2 \times EI}{{L_e}^2}, \text{ where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{2 \times \pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \mathbf{8224.5\ kN.} \quad \textbf{Ans.}$$

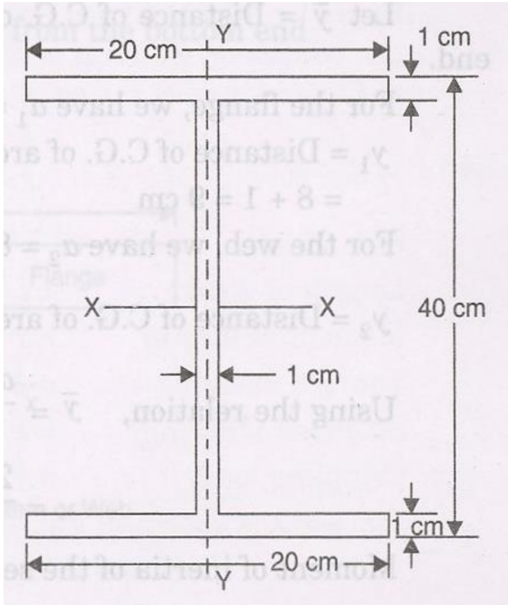
(ii) Crippling load when both the ends are pin-jointed

This is given by equation (19.1) in terms of actual length as

$$P = \frac{2\pi^2 \times EI}{l^2} \quad \text{where } l = \text{actual length} = 4000 \text{ mm}$$

$$= \frac{\pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = \mathbf{4112.25\ kN.} \quad \textbf{Ans.}$$

5.Determine Euler's crippling load for an I-section joist 40 cm x 20 cm x 1 cm and 5 m long which is used as a strut with both ends fixed. Take Young's modulus for the joist as 2.1 x 10^5 N/mm^2.



Sol. Given:

Dimensions of I-section = 40 cm x 20 cm x 1 cm

Length actual, l = 5 m = 5000 mm

Young's modulus, E = 2.1 x 10^5 N/mm^2.

Moment of inertia of the section about X-X axis,

I = M.O.I. of rectangle of dimension

20 cm x 40 cm - M.O.I. of rectangle of dimension

[(20 - 1), (40 – 1- 1)]

$$= \frac{1}{12} bd^3 - \frac{1}{12} b_1 d_1^3$$

$$= \frac{1}{12} \times 20 \times 40^3 - \frac{1}{12} \times 19 \times 38^3 \quad (\because \; b_1 = 19, d_1 = 38)$$

$$= 19786 \text{ cm}^4.$$

Similarly the moment of inertia of the section about Y-Y axis

I_{yy} = M.O.T. of rectangle of dimension (38 x 1)

+ M.O.I. of two rectangles of dimension (1 x 20)

$$= \frac{1}{12} \times 38 \times 1^3 + 2 \times \frac{1}{12} \times 1 \times 20^3$$

$$= 3.166 \times 1333.33 = 1336.5 \text{ cm}^4.$$

$\therefore$ Least value of the moment of inertia is about Y-Y axis.

$\therefore$ I = 1336.5 cm^4 = 1336.5 x 10^4 mm^4

As both the ends of the strut are fixed

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{5000}{2}$ = 2500 mm

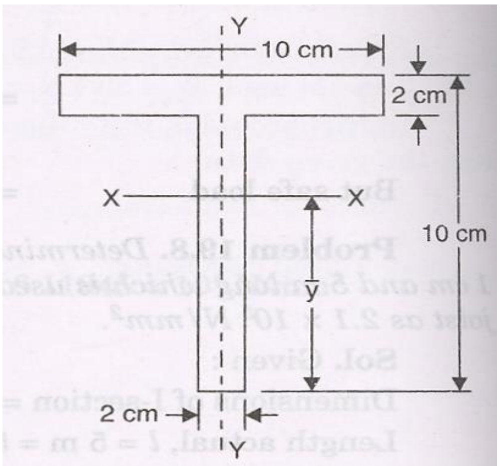
Let P = Crippling load.

Using the equation below, we get

$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.1 \times 10^5 \times 1336.5 \times 10^4}{2500^3}$$

$$= 4432080 \text{ N} = \mathbf{4.432 \text{ MN}}. \quad \textbf{Ans.}$$

6. Determine the crippling load for a T-section ofdimensions 10cm x 10 cm x 2 cm and of length 5 m when it is used as strut with both of its ends hinged. Take Young's modulus E $=2.0 \times 10^5 N/mm^2$.



Sol. Given :

Dimensions of T-section $= 10 \text{ cm} \times 10 \text{ cm} \times 2 \text{ cm}$

Length actual, $l = 5 \text{ m} = 5000 \text{ mm}$

Young's modulus, $E = 2.0 \times 10^5 \text{ N/mm}^2$.

First of all, calculate the C.G. of the section. The given section is symmetrical about the axis *Y-Y*, hence the C.G. of the section will lie on *Y-Y* axis.

Let $\bar{y}$ = Distance of C.G. of the section from bottom end.

For the flange, we have $a_1 = 10 \times 2 = 20 \text{ cm}^2$

y_1 = Distance of C.G. of area a_1 from the bottom end $= 8 + 1 = 9$ cm

For the web, we have $a_2 = 8 \times 2 = 16 \text{ cm}^2$

y_2 = Distance of C.G. of area a_2 from bottom end $= \frac{8}{2} = 4$ cm

Using the relation,
$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = 6.777 \text{ cm}$$

Moment of inertia of the section about the axis *X-X*,
$$I_{XX} = \left(\frac{10 \times 8^3}{12} + 20 \times 2.223^2\right) + \left(\frac{2 \times 8^3}{12} + 16 \times 2.777^2\right)$$
$$= (6.667 + 98.834) + (85.333 + 123.387) = 314.221 \text{ cm}^4.$$

Moment of inertia of the section about the axis *Y-Y*,
$$I_{YY} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12} = 166.67 + 5.33 = 172 \text{ cm}^4.$$

Least value of moment of inertia is about *Y-Y* axis

$\therefore$ $I = 172 \text{ cm}^4 = 172 \times 10^4 \text{ mm}^4$

Since the strut is hinged at both of its end

$\therefore$ Effect length, $L_e = l = 5000$ mm

Let P = Crippling load

Using equation (19.5), we get
$$P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 172 \times 10^4}{5000^3}$$ **= 135805.7 N. Ans.**

7. Calculate the Euler's critical load for a strut of T-section, the flange width being 10 cm, overall depth 8 cm and both flange and stem 1 cm thick. The strut is 3 m long and is built in at both ends. Take E = 2 x10^5 N/mm^5.

Sol. Given :

Actual length, l = 3 m = 3000 mm

Value of $E = 2 \times 10^5$ N/mm^2

The dimensions of T-section are shown in Fig. 19.10 (*a*). First of all, calculate the C.G. of the section. The given section is symmetrical about the *y-y* axis.

Hence the C.G. will lie on *y-y* axis.

Let $\bar{y}$ = Distance of C.G. of the section from the bottom end.

For the flange a_1 = Area of flange = 10 × 1 = 10 cm^2

y_1 = Distance of C.G. of area a_1 from the bottom end

$$= 7 + \frac{1}{2} = 7.5 \text{ cm.}$$

For the stem or web, a_2 = Area of stem = 7 × 1 = 7 cm^2

y_2 = Distance of C.G. of area a_2 from the bottom end

$$= \frac{7}{2} = 3.5 \text{ cm}$$

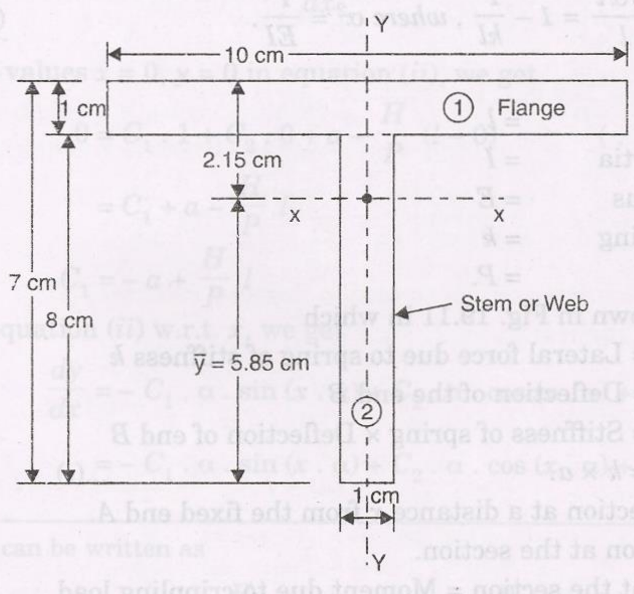


Fig. 19.10 (*a*)

Now using the relation, $\bar{y} = \dfrac{a_1 \times y_1 + a_2 \times y_2}{a_1 + a_2} = \dfrac{10 \times 7.5 + 7 \times 3.5}{(10+7)}$

$$= \frac{75 + 24.5}{17} = \frac{99.5}{17} = 5.85 \text{ cm.}$$

Now calculate the moment of inertia about *x-x* axis and *y-y* axis.

$$I_{XX} = \left[\frac{10 \times 1^3}{12} + a_1 \times (2.15 - 0.5)^2\right] + \left[\frac{1 \times 7^3}{12} + a_a \times (5.85 - 3.5)^2\right]$$

$$= \left[\frac{10}{12} + 10 \times 1.65^2\right] + \left[\frac{343}{12} + 7 \times 2.35^2\right] = 95.298 \text{ cm}^4$$

and $$I_{YY} = \frac{1 \times 10^3}{12} + \frac{7 \times 1^3}{12} = 83.33 + 0.583 = 83.916 \text{ cm}^4.$$

The least value of moment of inertia is about y-axis.

$$\therefore \quad I = I_{YY} = 83.916 \text{ cm}^4 = 839160 \text{ mm}^4.$$

As the strut is fixed at both ends, hence its effective length (L_e) will be half of its actual length.

$$\therefore \quad L_e = \frac{l}{2} = \frac{3000}{2} = 1500 \text{ mm}$$

Let P = Euler's critical load.

Using equation (19.5), $P = \dfrac{\pi^2 EI}{L_e^2} = \dfrac{\pi^2 \times 2 \times 10^5 \times 839160}{1500^3}$

$$= 736190 \text{ N} = \mathbf{736.19\ kN.} \quad \textbf{Ans.}$$

6.4 RANKINE'S FORMULA

we have learnt that Euler's formula gives correct results only for very long columns. But what happens when the column is a short or the column is not a very long. On the basis of results of experiments performed by Rankine, he established an empirical formula which is applicable to all columns whether they are short or long. The empirical formula given by

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \quad \ldots(i)$$

Rankine is known as Rankine's formula, which is given as

where P = Crippling load by Rankine's formula

P_c= Crushing load = x A

σ_c= Ultimate crushing stress

A = Area of cross-section

P_E= Crippling load by Euler's formula

$$= \frac{\pi^2 EI}{L_e^2}, \text{ in which } L_e = \text{Effective length}$$

For a given column material the crushing stress σ_c is a constant. Hence the crushing load P_c (which is equal to σ_c x A) will also be constant for a given cross-sectional area of the column. In equation (i) P_c is constant and hence value of P depends upon the value of P_E. But for a given column material and given cross-sectional area, the value of P_E depends upon the effective length of the column.

(*i*) *If the column is a short*, which means the value of L_e is small, then the value of P_E will be large. Hence the value of $\frac{1}{P_E}$ will be small enough and is negligible as compared to the value of $\frac{1}{P_C}$. Neglecting the value of $\frac{1}{P_E}$ in equation (*i*), we get

$$\frac{1}{P} \to \frac{1}{P_C} \text{ or } P \to P_C.$$

Hence the crippling load by Rankine's formula for a short column is approximately equal to crushing load. In Art. 19.2.1 also we have seen that short columns fail due to crushing.

(*ii*) *If the column is long*, which means the value of L_e is large. Then the value of P_E will be small and the value of $\frac{1}{P_E}$ will be large enough compared with $\frac{1}{P_C}$. Hence the value of $\frac{1}{P_C}$ may be neglected in equation (*i*).

$$\therefore \quad \frac{1}{P} = \frac{1}{P_E} \text{ or } P \to P_E.$$

Hence the crippling load by Rankine's formula for long columns is approximately equal to crippling load given by Euler's formula.

Hence the Rankine's formula $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$ gives satisfactory results for all lengths of columns, ranging from short to long columns.

Now the Rankine's formula is $\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C \cdot P_E}$.

Taking reciprocal to both sides, we have

$$P = \frac{P_C \cdot P_E}{P_E + P_C} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

(Dividing the numerator and denominator by P_E)

$$= \frac{\sigma_c \times A}{1 + \frac{\sigma_c \cdot A}{\left(\frac{\pi^2 EI}{L_e^2}\right)}} \qquad \left(\because P_c = \sigma_c \cdot A \text{ and } P_E = \frac{\pi^2 EI}{L_e^2}\right)$$

But $I = Ak^2$, where k = least radius of gyration

$\therefore$ The above equation becomes as

$$P = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot L_e^2}{\pi^2 E \cdot Ak^2}} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c}{\pi^2 E} \cdot \left(\frac{L_e}{k}\right)^2}$$

$$= \frac{\sigma_c \cdot A}{1 + a \cdot \left(\frac{L_e}{k}\right)^2}$$

where $a = \frac{\sigma_c}{\pi^2 E}$ and is known as Rankine's constant.

EXAMPLES

1. The external and internal diameter of a hollow cast iron column are 5cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of σ_c = 550 N/ mm^2 and

$a = \frac{1}{1600}$ in Rankine's formula.

Sol .Given:

External dia., D = 5 cm

Internal dia., d = 4 cm

$\therefore$ Area, $A = \frac{\pi}{4}(5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2$

Moment of Inertia, $I = \frac{\pi}{64}[5^4 - 4^4] = 5.7656\,\pi \text{ cm}^4$

$$= 5.7656\pi \times 10^4 \text{ mm}^4 = 57656\pi \text{ mm}^4$$

$\therefore$ Least radius of gyration,

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm}$$

Length of column, l = 3 m = 3000 mm

As both the ends are fixed,

$\therefore$ Effective length, $L_e = \frac{l}{2} = \frac{3000}{2} = 1500$ mm

Crushing stress, σ_c = 550 N/mm^2

Rankine's constant, $a = \frac{1}{1600}$

Let P = Crippling load by Rankine's formula

Using equation (19.9), we have

$$P = \frac{\sigma_c \cdot A}{1 + \left(\frac{L_e}{k}\right)^2} = \frac{550 \times 225\pi}{1 + \frac{1}{1600} \times \left(\frac{1500}{25.625}\right)^2}$$

$$= \frac{550 \times 225\pi}{3.1415} = \textbf{123750 N. Ans.}$$

$$= \frac{\pi}{64} \times 0.5904 \times D^4 = 0.009225 \times \pi \times D^4$$

But $I = A \times k^2$, where k is radius of gyration

$$\therefore \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.009225 \times \pi D^4}{\pi \times 0.09 \times D^2}} = 0.32D$$

Now using equation (19.9), $P = \dfrac{\sigma_c \cdot A}{1 + a\left(\dfrac{L_e}{k}\right)^2}$

$$\text{or} \quad 1250000 = \frac{550 \times \pi \times 0.09\, D^2}{1 + \dfrac{1}{1600} \times \left(\dfrac{2000}{0.32D}\right)^2} \qquad (\because \quad A = \pi \times 0.09D^2)$$

$$\frac{1250000}{550 \times \pi \times 0.09} = \frac{D^2}{1 + \dfrac{24414}{D^2}} \quad \text{or} \quad 8038 = \frac{D^2 \times D^2}{D^2 + 24414}$$

$$\text{or} \quad 8038D^2 + 8038 \times 24414 = D^4 \text{ or } D^4 - 8038D^2 - 8038 \times 24414 = 0$$

$$\text{or} \quad D^4 - 8038\, D^2 - 196239700 = 0.$$

2. A 1.5 m long column has a circular cross-section of 5 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using:

(a) Rankine's formula, take yield stress, σc = 560 N/mm2 and $a = \dfrac{1}{1600}$ for pinned ends.(b) Euler's formula, Young's modulus for C.I = 1.2 x 10^5 N/mm^2.

Sol. Given:

Length, 1 = 1.5 m = 1500mm

Diameter, d = 5 cm

The above equations is a quadratic equation in D^2. The solution is

$$\therefore \quad D^2 = \frac{8038 \pm \sqrt{8038^2 + 4 \times 1 \times 196239700}}{2} \qquad \left(\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{8038 \pm \sqrt{646094 + 784958800}}{2} = \frac{8038 \pm 29147}{2}$$

$$= \frac{8038 + 29147}{2} \quad \text{(The other root is not possible)}$$

$$= 18592.5 \text{ mm}^2$$

$$\therefore \quad D = \sqrt{18592.5} = 136.3 \text{ mm}$$

∴ External diameter = **136.3 mm. Ans.**

Internal diameter = 0.8 × 1363 = **109 mm. Ans.**

3. A short length of tube, 4cm internal diameter and 5cm external diameter, failed in compression at a load of 240 kN. When a 2 metre length of the same tube was tested as a strut with fixed ends, the load at failure was 158 kN. Assuming that σ_c in Rankine's formula is given by the first test, find the value of the constant a in the same formula. What will be the crippling load of this tube if it is used as a strut 3 m long with one end fixed and the other hinged ?

Sol. Given :

External diameter, $D = 5$ cm

Internal diameter, $d = 4$ cm

$\therefore$ Area, $$A = \frac{\pi}{4}(5^2 - 4^2) = \frac{9\pi}{4} = 2.25\,\pi\text{ cm}^2 = 225\,\pi\text{ mm}^2$$

Moment of inertia, $$I = \frac{\pi}{64}[5^4 - 4^4] = \frac{\pi}{64}(625 - 256)$$
$$= 5.7656 \times \pi\text{ cm}^4 = 57656\,\pi\text{ mm}^4$$

$\therefore$ Least radius of gyration, $$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\,\pi}{225\,\pi}} = 16\text{ mm}$$

Crushing load $= 240$ kN.

The value of σ_c in Rankine's formula is given by the crushing load of 240 kN.

$\therefore$ The value of $$\sigma_c = \frac{\text{Crushing load of 240 kN}}{\text{Area}}$$
$$= \frac{240}{225\,\pi} = 0.3395\text{ kN/mm}^2$$

Length of the strut, $l = 2$ m $= 2000$ mm

End condition = Both the ends are fixed

$\therefore$ Effective length, $$L_e = \frac{l}{2} = \frac{2000}{2} = 1000\text{ mm}$$

Crushing load of strut, $P = 158$ kN.

(*i*) *Value of constant 'a'*

Let a = Value of Rankine's constant

Using equation (19.9), we have

$$P = \frac{\sigma_c \, . \, A}{1 + a \, . \left(\frac{L_e}{k}\right)^2}$$

$$158 = \frac{0.33953 \times 225\,\pi}{1 + a \, . \left(\frac{1000}{16}\right)^2} = \frac{239.99}{1 + 3906.25 \times a}$$

$$1 + 3906.25 \times a = \frac{239.99}{158} = 1.5189.$$

$$\therefore \quad a = \frac{1.5189 - 1.0}{3906.25} = 0.0001328 = \frac{1}{7530}.$$ **Ans.**

(ii) Crippling load for the strut of length 3 m when one end is fixed and other is hinged

Actual length, l = 3 m = 3000 mm

End conditions = One end fixed and other is hinged

$\therefore$ Effective length, $L_e = \dfrac{l}{\sqrt{2}} = \dfrac{3000}{\sqrt{2}}$

Let P = Crippling load.

Using equation (19.9),

$$P = \frac{\sigma_c \cdot A}{1 + a.\left(\dfrac{L_e}{k}\right)^2}$$

$$= \frac{0.33953 \times 225\pi}{1 + \dfrac{1}{7530} \times \left(\dfrac{3000}{\sqrt{2} \times 16}\right)^2} \qquad \left(\because\ k = 16, a = \frac{1}{7530}\right)$$

$$= \frac{0.33953 \times 225 \times \pi}{1 + 2.3344} = \textbf{71.97 kN.}\quad \textbf{Ans.}$$

4.Two 300 mm x 120 mm I-section joists are united by 12 mm thick plates as shown in Fig. 1 to form a 7 m long stanchion. Given a factor of safety of 3, a compressive yield stress of 300 MN/m^2 and a constant a of 1/7500, determine the allowable load which can be carried by the stanchion according to the Rankine - Gordon formulae.

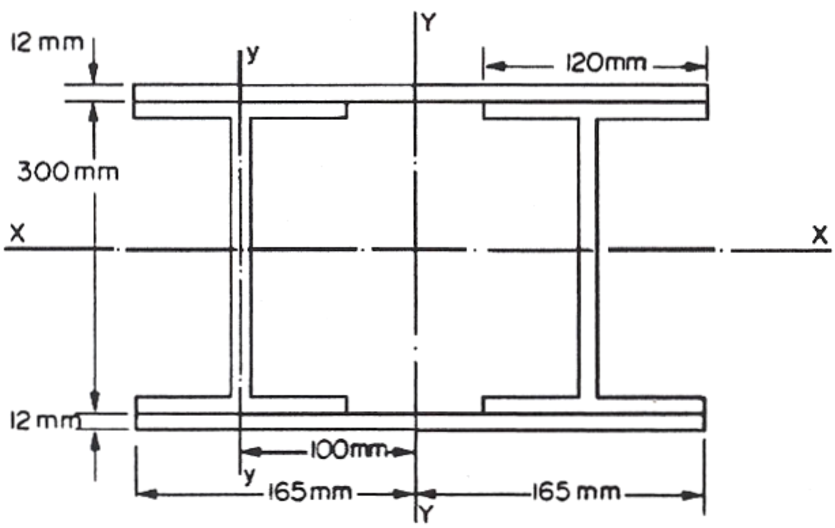


Fig. 1

The relevant properties of each joist are:

$$I_{xx} = 96 \times 10^{-6}\ \text{m}^4, \quad I_{yy} = 4.2 \times 10^{-6}\ \text{m}^4, \quad A = 6 \times 10^{-3}\ \text{m}^2$$

Solution

For the strut of Fig. 1:

From the parallel axis theorem:

$$I_{xx} \text{ for joists} = 2 \times 96 \times 10^{-6} = 192 \times 10^{-6}\ \text{m}^4$$

$$I_{xx} \text{ for plates} = 0.33 \times \frac{0.324^3}{12} - \frac{0.33 \times 0.300^3}{12}$$

$$= \frac{0.33}{12}[0.034 - 0.027] = 192.5 \times 10^{-6}\ \text{m}^4$$

$$\therefore \quad \text{total } I_{xx} = (192 + 192.5)10^{-6} = 384.5 \times 10^{-6}\ \text{m}^4$$

$$I_{yy} \text{ for joists} = 2(4.2 \times 10^{-6} + 6 \times 10^{-3} \times 0.1^2)$$

$$= 128.4 \times 10^{-6}\ \text{m}^4$$

and $$I_{yy} \text{ for plates} = 2 \times 0.012 \times \frac{0.33^3}{12} = 71.9 \times 10^{-6}\ \text{m}^4$$

$$\therefore \quad \text{total } I_{yy} = 200.3 \times 10^{-6}\ \text{m}^4$$

Now the smallest value of the Rankine-Gordon stress σ_R is given when k, and hence I, is a minimum.

$$\therefore \qquad \text{smallest } I = I_{yy} = 200.3 \times 10^{-6} = Ak^2$$

$$\text{total area } A = 2 \times 6 \times 10^{-3} + 2 \times 0.33 \times 12 \times 10^{-3} = 19.92 \times 10^{-3}$$

$$\therefore \qquad 19.92 \times 10^{-3} k^2 = 200.3 \times 10^{-6}$$

$$\therefore \qquad k^2 = \frac{200.3 \times 10^{-6}}{19.92 \times 10^{-3}} = 10.05 \times 10^{-3}$$

$$\therefore \qquad \left(\frac{L}{k}\right)^2 = \frac{7^2}{10.05 \times 10^{-3}} = 4.9 \times 10^3$$

and
$$\sigma_R = \frac{\sigma_y}{1 + a\left(\frac{L}{k}\right)^2} = \frac{300 \times 10^6}{1 + \frac{4.9 \times 10^3}{7500}}$$

$$\therefore \qquad = \frac{300 \times 10^6}{1.653} = 181.45 \text{ MN/m}^2$$

$$\therefore \qquad \text{allowable load} = \sigma_R \times A = 181.45 \times 10^6 \times 19.92 \times 10^{-3} = 3.61 \text{ MN}$$

With a factor of safety of 3 the maximum permissible load therefore becomes

$$P_{\max} = \frac{3.61 \times 10^6}{3} = \mathbf{1.203\ MN}$$

6.5 COLUMNS WITH ECCENTRIC LOAD

A column AB of length ‘l’ fixed at end A and free at end B. The column is subjected to a load P which is eccentric by an amount of ‘e’. The free end will sway sideways by an amount ‘a’ and the column will deflect as shown in Fig. (b).

Here a = deflection at free end B

e = Eccentricity

A = Area of cross-section of column

Consider any section at a distance x from the fixed end A.

Let y = deflection at the section then moment at the section = $P\,(a + e - y)$

(+ ve sign is taken due to sign convention given in Art. 19.4.1.)

But moment is also = $EI\dfrac{d^2y}{dx^2}$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = P\,(a + e - y) = P(a + e) - P \times y$$

or $$EI\frac{d^2y}{dx^2} + P \times y = P\,(a + e)$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \times y = \frac{P}{EI}\,(a + e)$$

The above equation can be written as

$$\frac{d^2y}{dx^2} + \alpha^2\,y = \alpha^2\,(a + e)$$

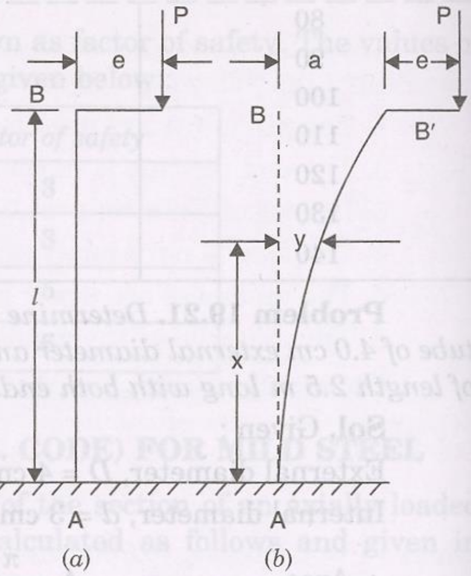


Fig. 19.13

where $$\alpha^2 = \frac{P}{EI} \quad \text{or} \quad \alpha = \sqrt{\frac{P}{EI}}$$

The complete solution of the above equation is,

$$y = C_1 \cos(\alpha \, . \, x) + C_2 \sin(\alpha \, . \, x) + (a + e)$$

$$= C_1 \cos\left(\sqrt{\frac{P}{EI}} \times x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}} \times x\right) + (a + e)$$

$$= C_1 \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x \times \sqrt{\frac{P}{EI}}\right) + (a + e) \quad ...(i)$$

and slope, $$\frac{dy}{dx} = -C_1 \left[\sin\left(x \times \sqrt{\frac{P}{EI}}\right)\right] \times \sqrt{\frac{P}{EI}} + C_2 \left[\cos\left(x \times \sqrt{\frac{P}{EI}}\right)\right] \times \sqrt{\frac{P}{EI}} + 0 \quad ...(ii)$$

($\because$ $a + e$ is constant, Hence differential is zero)

In equations (*i*) and (*ii*) C_1 and C_2 are constant of integration. Their values are obtained from boundary conditions.

(*i*) At A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$ ($\because$ A is a fixed end)

From equation (*i*) where $x = 0$ and $y = 0$, we get

$$0 = C_1 + a + e \quad \therefore \quad C_1 = -(a + e)$$

From equation (*ii*) where $x = 0$ and $\frac{dy}{dx} = 0$, we get

$$0 = C_2 \times \sqrt{\frac{P}{EI}} \quad \therefore \quad C_2 = 0 \qquad \left(\because \frac{P}{EI} \text{ can not be zero}\right)$$

Substituting the values of C_1 and C_2 in equation (*i*), we get

$$y = -(a + e) \cos\left(x \times \sqrt{\frac{P}{EI}}\right) + (a + e) \quad ...(iii)$$

Maximum stress

Let us find the maximum compressive stress for the column section. Due to eccentricity, there will be bending stress and also direct stress.

$\therefore \quad \sigma_{max} = \sigma_0 + \sigma_b$ where σ_0 = direct stress $= \dfrac{P}{A}$

σ_b = Max. bending stress.

The maximum bending stress will be at the section where bending moment is maximum. Bending moment is maximum at the fixed end A.

$\therefore \quad$ Max B.M. at A, $M = P \times (a + e)$

$$= P \times e \sec\left(l \times \sqrt{\frac{P}{EI}}\right) \quad \left[\because \ a + e = e \sec\left(l \times \sqrt{\frac{P}{EI}}\right) \text{ from equation } (iv)\right]$$

Using $\quad \dfrac{M}{I} = \dfrac{\sigma_b}{y} \quad$ or $\quad \sigma_b = \dfrac{M}{I} \times y = \dfrac{M}{\left(\dfrac{I}{y}\right)}$

$$= \frac{M}{Z} \quad \text{where } Z = \frac{I}{y} = \text{Section modulus}$$

$$= \frac{P \times e \sec\left(l \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \quad \left[\because \ M = P \times e \sec\left(l \times \sqrt{\frac{P}{EI}}\right)\right]$$

Hence maximum compressive stress becomes as

$$\therefore \quad \sigma_{max} = \sigma_0 + \sigma_b = \frac{P}{A} + \frac{P \times e \sec\left(l \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \qquad \text{...(19.12)}$$

The equation (19.12) is used for a column whose one end is fixed, other end is free and load is eccentric to the column. In this equation, l is the actual length of the column. The relation between actual length and effective length for a column whose one end is fixed and other end is free is given by

$$L_e = 2l \quad \text{or} \quad l = \frac{L_e}{2}$$

Substituting the value of l is equation (19.12), we get a general formula which can be used for any end condition. Hence general formula is

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\dfrac{L_e}{2} \times \sqrt{\dfrac{P}{EI}}\right)}{Z} \qquad \text{...(19.13)}$$

(ii) at $x = l$, $y = a$, hence equation (iii) becomes as

$$a = -(a+e)\cos\left(l \times \sqrt{\frac{p}{EI}}\right) + (a+e)$$

or $$(a+e)\cos\left(l \times \sqrt{\frac{P}{EI}}\right) = a + e - a = e$$

$$\therefore \quad a + e = \frac{e}{\cos\left(l \times \sqrt{\frac{P}{EI}}\right)} = e \sec\left(l \times \sqrt{\frac{P}{EI}}\right) \quad ...(iv)$$

EXAMPLES

1. A column of circular section is subjected to a load of 120 kN. The load is parallel to the axis but eccentric by an amount of 2.5 mm. The external and internal diameters of columns are 60mm and 50mm respectively. If both the ends of the column are hinged and column is 2.1 m long, then determine the maximum stress in the column. Take E = 200 GN/m^2.

Sol. Given:

Load, P = 120 kN = 120 x10^3 N

Eccentricity, e = 2.5 mm = 2.5 x 10^{-3} m

D = 60mm = 0.06 m, d = 50mm = 0.05 mm, l = 2.1 m

Both ends are hinged, Le = l = 2.1 m

Value of E = 200 GN/m^2 = 200 x 10^9 N/m^2

The maximum stress is given by equation (19.13) as

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z} \quad ...(i)$$

where A = Area of section

$$= \frac{\pi}{4}[D^2 - d^2] = \frac{\pi}{4}[0.06^2 - 0.05^2]$$

$$= \frac{\pi}{4} \times 0.0011 = 8.639 \times 10^{-4} \text{ m}^2$$

$$I = \text{Moment of inertia} = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{64}(0.06^4 - 0.05^4) \text{ mm}^4$$

$$= \frac{\pi}{64}(1.296 \times 10^{-5} - 0.625 \times 10^{-5}) = 0.0329 \times 10^{-5}$$

Z = Section modulus

$$= \frac{I}{y} = \frac{\pi}{64}\frac{[D^4 - d^4]}{\left(\frac{D}{2}\right)} = \frac{\frac{\pi}{64}[0.06^4 - 0.05^4]}{0.03}$$

$$= \frac{\pi(1.296 \times 10^{-5} - 0.625 \times 10^{-5})}{64 \times 0.03}$$

$$= 1.0975 \times 10^{-5} \text{ m}^3$$

2. If the given column of problem 1 is subjected to an eccentric load of 100 kN and maximum permissible stress is limited to 320 MN/m^2, then determine the maxi mum eccentricity of the load.

Sol. Given:

Data from problem 1

D = 60mm = 0.06 m, d = 50mm = 0.05 m, l = 2.1 m, Le = l = 2.1 m,

E = 200 GN/m^2 = 200 x 10 N/m^2., I = 0.0329 x10^{-5} m^4,

Z = 1.0975 x10^{-5} m^3, A = 8.639 x$10^{-4}$$m^2$

Eccentric load, P = 100 kN = 100 x 10^3 N

Max. stress, σ_{max}= 320 MN/m^2 = 320 x 10^6 N/m^2

Let e = Maximum eccentricity

Using equation (19.13), we get

$$\sigma_{max} = \frac{P}{A} + \frac{P \times e \times \sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)}{Z} \quad \ldots(i)$$

Let us first find the value of sec $\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right)$.

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left[\frac{2.1}{2} \times \sqrt{\frac{100 \times 10^3}{200 \times 10^9 \times 0.0329 \times 10^{-5}}}\right]$$

$$= \sec[1.294 \text{ rads}] = \sec\left(1.294 \times \frac{180^0}{\pi}\right)$$

$$= \sec(74.16^0) = 3.665$$

Substituting the known values in equation (*i*), we get

$$320 \times 10^6 = \frac{100 \times 10^3}{8.639 \times 10^{-4}} + \frac{(100 \times 10^3) \times e \times 3.665}{1.0975 \times 10^{-5}}$$

$$= 115.754 \times 10^6 + 33394\, e \times 10^6$$

or $$320 = 115.754 + 33394 \times e$$

or $$e = \frac{320 - 115.754}{33394} \text{ m} = 6.116 \times 10^3 \text{ m} = \mathbf{6.116 \text{ mm.}} \quad \textbf{Ans.}$$

$$\sec\left(\frac{L_e}{2} \times \sqrt{\frac{P}{EI}}\right) = \sec\left(\frac{2.1}{2} \times \sqrt{\frac{120 \times 10^3}{200 \times 10^9 \times 0.329 \times 10^{-5}}}\right)$$

$= \sec$ (1.4179 radians) (Here 1.4179 is in radians $= 1.4179 \times \frac{180}{\pi} = 81.239$)

$$= \sec(81.239) = 6.566$$

Substituting these values in equation (*i*) above, we get

$$\sigma_{max} = \frac{120 \times 10^3}{8.639 \times 10^{-4}} + \frac{(120 \times 10^3) \times (2.5 \times 10^{-3}) \times 6.566}{1.0975 \times 10^{-5}}$$

$$= 138.9 \times 10^6 + 179.48 \times 10^6 \text{ N/m}^2$$

$$= \mathbf{318.38 \times 10^6 \text{ N/m}^2 \quad or \quad 318.38 \text{ N/mm}^2} \quad \textbf{Ans.}$$